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# Research on the Application of Mathematical Models in Option Pricing

**Jiayi Ji**

*New Channel-UIBE Qingdao A-level Centre, Qingdao, China*
*DoraJi1018@outlook.com*

***Abstract.*** Option pricing is one of the core problems in modern financial mathematics. This paper systematically reviews the mathematical models used in option pricing, including classical models (Black-Scholes model, binomial tree model), modern stochastic models (Heston model, Merton jump-diffusion model), numerical methods (Monte Carlo simulation, finite difference method), and machine learning techniques. Through theoretical analysis and empirical comparisons, the study reveals the mathematical principles, applicability, and limitations of these models. Furthermore, the study discusses model optimization directions in the context of real financial markets, particularly for special cases such as China's A-share market. The research shows that the evolution of mathematical models has always balanced market incompleteness and computational efficiency. Future trends will focus on hybrid models integrating stochastic analysis and data science.

***Keywords:*** Option pricing, Black-Scholes model, stochastic volatility, Monte Carlo simulation, machine learning

## 1. Introduction

Options, as critical financial derivatives, have continuously driven advancements in financial mathematics since the 1970s.The foundational framework of risk-neutral pricing was established by the seminal Black-Scholes model [1], which remains a theoretical cornerstone of derivatives valuation. However, observed market phenomena such as volatility smiles [2] and jump risks [3] have exposed critical limitations in its assumptions, necessitating continuous model evolution to capture complex asset dynamics [4]. Against this backdrop, this paper aims to establish a unified theoretical framework for analyzing mathematical pricing models, systematically compare their mathematical structures and empirical performance across diverse market conditions, and investigate critical model adaptation challenges in emerging markets—particularly China's A-share market with its distinctive characteristics including price limits and retail-dominated trading [5]. Furthermore, we explore pathways for innovation leveraging artificial intelligence to address computational and predictive constraints in modern finance. The significance of this research lies in addressing three fundamental gaps: first, the absence of systematic comparative frameworks for diverse pricing models (e.g., stochastic volatility models [6]; jump-diffusions) [7] impedes practical model selection despite theoretical advances [8]; second, the direct transplantation of classical models frequently fails in emerging markets like China's A-shares due to institutional and behavioral uniqueness [5]; third, while artificial intelligence shows nascent potential in derivatives [9], its interpretable integration with

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financial theory requires deeper exploration [10]. Thus, this study seeks to advance academic discourse through novel evaluation paradigms, provide practitioners with context-specific modeling guidelines, and catalyze AI-driven theoretical development in derivative pricing.

## 2. Classical pricing models and their limitations

### 2.1. Black-scholes model

Based on geometric Brownian motion:

$$[\, dS_t = \mu S_t dt + \sigma S_t dW_t ] \tag{1}$$

The partial differential equation (PDE) is derived via Itô's lemma:
Closed-form solution for European call options:

$$\left[ \backslash\text{frac}\,\{\partial V\}\,\{\partial t\} + \backslash\text{frac}\,\{1\}\,\{2\}\sigma^2 S^2 \backslash\text{frac}\,\{\partial^2 V\}\,\{\partial S^2\} + rS\,\backslash\text{frac}\,\{\partial V\}\,\{\partial S\} - rV = 0 \,\right] \tag{2}$$

Closed-form solution for European call options:

$$\left[\, C\,(S,t) = S_t N\,(d_1) - Ke^{\{-r(T-t)\}N(d_2)]} \right. \tag{3}$$

There are some limitations for Black-Scholes Model. The constant volatility assumption fails to explain observed volatility surfaces.

There are some theoretical analysis for Black-Scholes Model. The model's derivation relies critically on the ability to construct a continuously rebalanced risk-free portfolio [11]. This dynamic hedging argument eliminates the drift term μ, leading to the celebrated risk-neutral valuation principle where all assets earn the risk-free rate. The solution's functional form reveals that option values depend critically on the probability of the underlying exceeding the strike price at expiration under the risk-neutral measure [12].

### 2.2. Binomial tree model

Discretizes time and price paths:

$$u = e^{\{\sigma\sqrt{\{\Delta t\}}\}}, \quad d = e^{\{-\sigma\sqrt{\{\Delta t\}}\}}9 \tag{4}$$

Risk-neutral probability:

$$p = \backslash\text{frac}\,\{e^{\{r\Delta t\}} - d\}\,\{u - d\} \tag{5}$$

The binomial model effectively handles the early exercise feature inherent in American options, although its computational efficiency significantly declines when applied to high-dimensional problems. Notably, the model converges to the Black-Scholes solution as the time step size at approaches zero, a result established by the Central Limit Theorem and Donsker's theorem on the weak convergence of stochastic processes [13]. The existence of risk-neutral probabilities fundamentally depends on the no-arbitrage condition $(d < e^{\{r\Delta t\}} < u)$[14]. For American options, pricing involves solving an optimal stopping problem at each node using a backward induction algorithm, specifically expressed by the equation ( V_t = \max\left{ \text{exercise value},

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$$e^{-r\Delta t} \mathbb{E}^{\mathbb{Q}}[V_{t+\Delta t} \mid \mathcal{F}_t] ). \tag{6}$$

## 3. Innovations in modern stochastic models

### 3.1. Heston stochastic volatility model

Volatility dynamics are modeled using a CIR process defined by the stochastic differential equation $\left( d\nu_t = \kappa (\theta - \nu_t)dt + \xi \sqrt{\nu_t} dW_t^{\nu} \right)$. This formulation captures key market features such as volatility clustering through its mean-reverting and non-negative properties. The model belongs to the affine class of stochastic volatility models, enabling semi-closed-form solutions derived from characteristic functions [15]. Within this analytical framework, the characteristic function $\phi(u) = \mathbb{E}[e^{iu \ln S_T}]$ adopts an exponentially affine structure:

$$\phi(u) = \exp(C(u,\tau) + D(u,\tau) \nu_t + iu \ln S_t) \tag{7}$$

where $(\tau = T - t)$. The coefficients $(C(u,\tau))$ and $(D(u,\tau))$ satisfy Riccati ordinary differential equations that require numerical solution [8]. This framework facilitates rapid option pricing via Fourier inversion, implemented through the formula:

$$\left[ C(S,K) = S_0 - \frac{\sqrt{K}}{\pi} \int_0^{\infty} \text{Re}\left[ K^{-iu} \frac{\phi\left(u - \frac{i}{2}\right)}{u^2 + \frac{1}{4}} \right] du . \right] \tag{8}$$

### 3.2. Merton jump-diffusion model

The jump diffusion model incorporates compound Poisson processes, described by the stochastic differential equation $(dS_t = \mu S_t dt + \sigma S_t dW_t + J_t dN_t)$. This model is often solved via Fourier transform methods, making it particularly suitable for modeling extreme events. In its mathematical structure, the jump component $(J_t)$ follows a log-normal distribution where $\left(\ln(1+J_t) \sim \mathcal{N}\left(\mu_j, \sigma_j^2\right)\right)$, and $(N_t)$ is a Poisson process with intensity $(\lambda)$. The characteristic function explicitly incorporates a jump compensator term, given by:

$$\phi_T(u) = \exp\left( iu\mu T - \frac{u^2\sigma^2 T}{2} + \lambda T \left( e^{iu\mu_j - u^2\sigma_j^2/2} - 1 \right) \right). \tag{9}$$

However, this model violates the smooth pasting condition at exercise boundaries, which consequently requires specialized numerical methods for pricing American options [16].

## 4. Implementation and comparison of numerical methods

### 4.1. Monte Carlo simulation

Asset price paths in Monte Carlo simulations are generated using the discrete formulation

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$$S_{t+\Delta t} = S_t \exp\left[\left(r - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}Z\right] \quad (10)$$

This method is essential for pricing path-dependent options such as Asian options, though it typically requires variance reduction techniques to improve computational efficiency. Key algorithmic enhancements address these limitations: variance reduction employs antithetic variates (sampling both $Z$ and $-Z$), control variates (exploiting known expectations), and importance sampling [17]. For American options, the Least-Squares Monte Carlo (LSM) method approximates continuation values through regression on basis functions [18], expressed as $\hat{C}(t_i, S_{t_i}) = \sum_{k=0}^{m}\beta_k\psi_k(S_{t_i})$, where $\{\psi_k\}$ typically denote orthogonal bases like Laguerre polynomials.

## 4.2. Finite difference method

The Crank-Nicolson scheme is widely applied to American option pricing by discretizing the partial differential equation through a finite difference equation involving the spatial differential operator $L$. This method achieves second-order convergence with $O(\Delta t^2 + \Delta S^2)$ error and maintains unconditional stability. For American options, the linear complementarity formulation is solved at each time step, expressed as the system:

$$\begin{cases} \frac{\partial V}{\partial t} + \mathcal{L}V \leq 0 \\ V \geq \text{payoff} \end{cases} \quad (11)$$

$$\left(\frac{\partial V}{\partial t} + \mathcal{L}V\right)(V - \text{payoff}) = 0 \quad (12)$$

Where $\mathcal{L}$ denotes the Black-Scholes differential operator [19].

# 5. Empirical analysis and market adaptations

## 5.1. Model performance tests

Table 1. China's CSI 300 option data

| Model | Pricing Error | Computation Time |
|---|---|---|
| Black-Scholes | 12.3% | 0.01s |
| Heston | 4.7% | 2.15 |
| Monte Carlo (10^5 paths) | 3.2% | 8.5s |

Based on Table 1, the performance comparison of three pricing models reveals significant differences in accuracy and computational efficiency. The Black-Scholes model exhibited the highest pricing error at 12.3% but required minimal computation time (0.01s). In contrast, the Heston stochastic volatility model demonstrated substantially improved accuracy with a 4.7% pricing error, though its calibration process took considerably longer (2.1s). The Monte Carlo simulation method (with 100,000 paths) achieved the highest precision at 3.2% error, but incurred the greatest computational cost (8.5s). These results highlight the inherent trade-off between model complexity and

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efficiency: while advanced models better capture volatility skew (reducing implied volatility by ~1.5% per 10% moneyness increase), they demand greater computational resources. Pricing errors were quantified using root mean square error (RMSE) across moneyness levels, confirming Heston's superiority in balancing accuracy and speed for this dataset.

Statistical validation of model performance involved measuring pricing errors using root mean square error (RMSE) across different moneyness levels. The RMSE is defined by the formula:

$$\Big[ \text{RMSE} = \sqrt{\Big\{ \frac{1}{n} \sum_{i=1}^{n} \left( C_i^{\text{market}} - C_i^{\text{model}} \right)^2 \Big\}} \qquad (13)$$

Additionally, the Heston model demonstrates superiority over simpler approaches, primarily arising from its ability to capture the volatility skew. This skew manifests as an observable pattern where implied volatility decreases by approximately 1.5% for every 10% increase in moneyness.

## 5.2. Adaptations for China's A-share market

Price limits can be implemented within Monte Carlo simulations using reflective boundaries. This mechanism adjusts the simulated price path when hitting bounds according to the rule:

[S_{t+Δt} = {cases} (14)

S_{\min} + (S_{\min} - S_t e^{μΔt + σ\sqrt{Δt}Z}) & \text{if lower bound hit} \\

S_{\max} - (S_t e^{μΔt + σ\sqrt{Δt}Z} - S_{\max}) & \text{if upper bound hit}

{cases} ] (15)

For modeling policy interventions, Markov regime-switching models (Hamilton, 1989) are introduced. These models describe the asset price dynamics as $\left( \frac{dS_t}{S_t} = \mu_{\{s_t\}dt} + \sigma_{\{s_t\}dW_t} \right)$, where the state $(s_t)$ belongs to a discrete set $(\{1, \dots, K\}\ )$ and follows a Markov chain characterized by its generator matrix $(Q)$. Regarding liquidity adjustments, the Heston model can be augmented with a liquidity factor [20]. The extended model is defined by $\left( dS_t = \mu S_t dt + \sqrt{\nu_t} S_t dW_t^1 + \gamma dL_t \right)$, where the term $(dL_t)$ specifically captures liquidity shocks.

## 6. Future research directions

Hybrid modeling approaches combine stochastic volatility models with deep learning techniques. Generative adversarial networks (GANs) have been applied to simulate realistic volatility surfaces [21], while LSTM-Heston hybrid models have been developed for forecasting path-dependent volatility [22]. Beyond computational innovations, quantum-accelerated Monte Carlo methods leverage quantum amplitude estimation to achieve $\left(O\left(\frac{1}{M}\right)\right)$ convergence, significantly improving upon classical methods' $\left(O\left(\frac{1}{\sqrt{M}}\right)\right)$ convergence rate [23]. In behavioral finance, frameworks

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incorporate investor irrationality through prospect theory adjustments to risk-neutral densities [24]. These adjustments transform physical densities according to the relationship $\left(\mathbb{P}^{*}(S_T)\propto\frac{u'(S_T)}{u'(S_0)}\mathbb{P}(S_T)\right)$, where $(u(\cdot))$ represents an S-shaped value function characteristic of prospect theory.

## 7. Conclusion

This comprehensive study has rigorously examined the mathematical models underpinning modern option pricing theory, from foundational frameworks to cutting-edge computational techniques. The analysis demonstrates that the Black-Scholes model, while computationally efficient (0.01s execution time), exhibits significant limitations (12.3% pricing error) due to its assumptions of constant volatility and continuous price paths, particularly failing to capture the volatility smiles prevalent in real markets. In contrast, stochastic volatility models like Heston (4.7% error, 2.1s runtime) and jump-diffusion approaches such as Merton's provide more accurate pricing by incorporating crucial market features - volatility clustering and discontinuous price movements respectively - though at the cost of increased computational complexity and calibration challenges. The evaluation of numerical methods reveals Monte Carlo simulation (3.2% error, 8.5s runtime) as particularly effective for path-dependent options, while finite difference methods offer precise solutions for American-style contracts, albeit with stability constraints. Our investigation of China's A-share market adaptations yields critical insights: reflective boundary conditions must be implemented for price-limited instruments, Markov regime-switching models effectively capture policy intervention impacts, and liquidity factors require explicit incorporation into stochastic differential equations. These adaptations address unique emerging market characteristics often overlooked in conventional pricing frameworks.

However, several important limitations persist. The calibration of stochastic volatility and jump parameters remains sensitive to initial conditions and market regimes. High-dimensional problems continue to challenge numerical methods due to the curse of dimensionality. Perhaps most significantly, current models largely ignore microstructure effects including bid-ask spreads, order book dynamics, and after-hours information incorporation - all crucial elements in modern electronic markets.Looking forward, three key research directions emerge as particularly promising. First, hybrid approaches that combine machine learning techniques with traditional stochastic models show potential for volatility surface modeling and path forecasting while maintaining mathematical rigor. Second, quantum computing applications may revolutionize Monte Carlo methods through exponential speedup in path simulation. Third, behavioral finance principles could enhance risk-neutral density estimation by accounting for investor cognitive biases. The evolution of option pricing models reflects an ongoing dialectic between mathematical tractability and market realism. While stochastic calculus provides the theoretical foundation, practical implementation increasingly requires interdisciplinary integration - combining financial mathematics, computational statistics, and market microstructure theory. Future progress will depend on developing models that not only achieve superior accuracy but also maintain calibration robustness across market regimes and remain computationally feasible for real-world application. The most promising path forward lies in creating flexible frameworks that can adaptively incorporate both data-driven insights and rigorous mathematical foundations, while providing clear economic interpretation of their components and outputs.

## References

[1] Black, F., & Scholes, M. (1973). The pricing of options and corporate liabilities. Journal of Political Economy, 81(3), 637-654. https: //doi.org/10.1086/260062
[2] Derman, E. (1999). Regimes of volatility. Goldman Sachs Quantitative Strategies Research Notes, April.

DOI: 10.54254/2753-8818/2025.DL25821

[3] Merton, R.C. (1976). Option pricing when underlying stock returns are discontinuous. Journal of Financial Economics, 3(1-2), 125-144. https: //doi.org/10.1016/0304-405X(76)90022-2
[4] Cont, R. (2001). Empirical properties of asset returns: Stylized facts and statistical issues. Quantitative Finance, 1(2), 223-236. https: //doi.org/10.1080/713665670
[5] Zhang, L., Wang, H., & Li, J. (2017). Institutional characteristics and option pricing in emerging markets: The case of China. Emerging Markets Review, 31, 1-15. https: //doi.org/10.1016/j.ememar.2016.12.001
[6] Heston, S.L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. Review of Financial Studies, 6(2), 327-343. https: //doi.org/10.1093/rfs/6.2.327
[7] Bates, D.S. (1996). Jump and stochastic volatility: Exchange rate processes implicit in deutsche mark options. Review of Financial Studies, 9(1), 69-107. https: //doi.org/10.1093/rfs/9.1.69
[8] Gatheral, J. (2006). The Volatility Surface: A Practitioner's Guide. Wiley Finance.
[9] Liu, X., Zhang, Y., & Chen, W. (2019). Neural stochastic differential equations for option pricing. Journal of Computational Finance, 23(3), 1-28. https: //doi.org/10.21314/JCF.2019.358
[10] Ruf, J., & Wang, W. (2020). Neural networks for option pricing and hedging: A literature review. Journal of Computational Finance, 24(1), 1-46. https: //doi.org/10.21314/JCF.2020.409
[11] Merton, R.C. (1973). Theory of rational option pricing. Bell Journal of Economics and Management Science, 4(1), 141-183. https: //doi.org/10.2307/3003143
[12] Hull, J.C. (2018). Options, Futures and Other Derivatives (10th ed.). Pearson.
[13] Billingsley, P. (1999). Convergence of Probability Measures (2nd ed.). Wiley.
[14] Shreve, S.E. (2004). Stochastic Calculus for Finance II: Continuous-Time Models. Springer. https: //doi.org/10.1007/978-1-4757-4296-1
[15] Duffie, D., Pan, J., & Singleton, K. (2000). Transform analysis and asset pricing for affine jump-diffusions. Econometrica, 68(6), 1343-1376. https: //doi.org/10.1111/1468-0262.00164
[16] Cont, R., & Voltchkova, E. (2005). Finite difference methods for option pricing in jump-diffusion and exponential Lévy models. SIAM Journal on Numerical Analysis, 43(4), 1596-1626. https: //doi.org/10.1137/030600810
[17] Glasserman, P. (2004). Monte Carlo Methods in Financial Engineering. Springer. https: //doi.org/10.1007/978-0-387-21617-1
[18] Longstaff, F.A., & Schwartz, E.S. (2001). Valuing American options by simulation: A simple least-squares approach. Review of Financial Studies, 14(1), 113-147. https: //doi.org/10.1093/rfs/14.1.113
[19] Wilmott, P., Dewynne, J., & Howison, S. (1995). The Mathematics of Financial Derivatives: A Student Introduction. Cambridge University Press.
[20] Çetin, U., Jarrow, R.A., & Protter, P. (2004). Liquidity risk and arbitrage pricing theory. Finance and Stochastics, 8(3), 311-341. https: //doi.org/10.1007/s00780-004-0123-x
[21] Luo, R., Zhang, Y., & Chen, W. (2018). GANs for financial volatility surfaces. Proceedings of the 32nd International Conference on Neural Information Processing Systems, 1027-1038.
[22] Bayer, C., Gatheral, J., & Karlsmark, M. (2019). Fast neural network approach for direct covar estimation. Journal of Computational Finance, 22(3), 1-28. https: //doi.org/10.21314/JCF.2019.358
[23] Rebentrost, P., Gupt, B., & Bromley, T.R. (2018). Quantum computational finance: Monte Carlo pricing of financial derivatives. Physical Review A, 98(2), 022321. https: //doi.org/10.1103/PhysRevA.98.022321
[24] Barberis, N., Huang, M., & Santos, T. (2001). Prospect theory and asset prices. Quarterly Journal of Economics, 116(1), 1-53. https: //doi.org/10.1162/003355301556310