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Theory and Review of Gravitational Waves Induced by Scalar Perturbations in Second-order

Zichen Zhang

School of Physics and Astronomy, College of Science and Engineering, University of Glasgow, Glasgow, United Kingdom
1722384756@qq.com

Abstract. Cosmological perturbation theory serves as a fundamental approach for exploring the development of cosmic structures and the dynamics of the early universe. While linear (first-order) perturbation theory has been remarkably successful in explaining phenomena such as the cosmic microwave background (CMB) and the large-scale structure (LSS), second-order effects—especially those arising from nonlinear interactions—cannot be neglected. Notably, second-order scalar perturbations have the capacity to generate gravitational waves via nonlinear couplings, offering additional insight into the early universe. This study focuses on the theoretical formulation of such gravitational waves, detailing the perturbative expansion in Newton gauge, the formulation of source terms for second-order modes, and the application of the transverse-traceless (TT) decomposition technique.

Keywords: Scalar Perturbation, Gravitational Wave, Second Order, Ricci Tensor, Christoffel Symbol

1. Introduction

In cosmology, the theory of cosmic perturbations is an important foundation for studying the formation of cosmic structures and the dynamics of the early universe. The standard cosmological model holds that cosmic structures originate from tiny density fluctuations on a uniform background. These fluctuations gradually grow through gravity and eventually form large-scale structures such as galaxies [1]. Today's observations (such as CMB) verify this process. However, with the continuous improvement of observation accuracy, especially in the field of gravitational wave observations, researchers have found that ignoring higher-order nonlinear perturbation effects may miss important physical information [2].This paper focuses on second-order scalar perturbations, but the principles of higher-order perturbations are similar.

Although there already exist many observational and theoretical models regarding scalar perturbation-induced gravitational waves, there is still a lack of systematic, clear, and accessible explanations for the formation mechanism of this phenomenon.

In the first-order perturbation theory, scalar, vector and tensor perturbations are decoupled from each other, and scalar (density perturbation), vector (vorticity) and tensor (gravitational waves) propagate independently. However, in the second-order perturbation theory, the situation becomes

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more complicated: even if there are only scalar perturbations at first, due to the nonlinear characteristics of the gravitational field equations, gravitational waves will be generated at the second order [3]. This effect is called scalar perturbation-induced gravitational waves. Unlike the primordial gravitational waves, this type of gravitational wave is not a "primordial ripple" left over from the inflation period, but a "secondary ripple" caused by the uneven distribution of matter in the later period.

In this paper, we will explore how second-order scalar perturbations appear explicitly in the Einstein equations in the form of scalar potential functions ϕ and ψ under the Newtonian gauge, and further explore how to use the transverse-traceless projection (TT projection) method to rigorously extract the gravitational wave component from general second-order perturbations.

2. Background

2.1. Perturbation theory

Perturbation theory analyzes problems by decomposing the physical quantities of a system into background quantities and small perturbations. In cosmology, this method is widely used in the evolution analysis of cosmic structures. Specifically, it is to use Taylor expansion to this physical quantity under a certain background [4].

For example, the physical quantity Q can be expressed as:

$$Q = Q^{(0)} + \delta Q^{(1)} + \frac{1}{2}\delta Q^{(2)} + \ldots$$

Where $Q^{(0)}$ is the background value, $\delta Q^{(1)}$ and $\delta Q^{(2)}$ is the first-order and second-order perturbation terms respectively.

Therefore, the perturbation expansion of metric is:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h^{(1)}_{\mu\nu} + h^{(2)}_{\mu\nu} + \cdots$$

and the contravariant is:

$$g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{(1)\mu\nu} - h^{(2)\mu\nu} + h^{(1)\mu\alpha}h^{(1)\nu}_{\alpha} + \ldots [5]$$

Friedmann–Lemaître–Robertson–Walker(FLRW) Metric and Newtonian Gauge

The FLRW Metric is a metric that describes a uniform, isotropic, expanding (or contracting) universe. Its standard form is as follows:

$$ds^2 = -dt^2 + a^2(t)\,\delta_{ij}\,dx^i dx^j$$

where a(t) is the cosmological scale factor, describing the expansion of the universe; δ_{ij} is the Kronecker delta of the space part, indicating that the space is isotropic and flat; dt and dxidxj are the time and space micro-elements, respectively [6].

However, when using general relativity, many redundant degrees of freedom appear. By choosing a specific gauge (such as the Newtonian gauge), these redundancies can be avoided effectively, making the calculations more concise and clear [7].

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We use the Newtonian metric.Under this metric, the metric becomes:

$$ds^2 = -(1+2\phi)\,dt^2 + a^2(t)\,(1-2\psi)\delta_{ij}\,dx^i\,dx^j$$

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{0j} \\ g_{i0} & g_{ij} \end{pmatrix} = \begin{pmatrix} -(1+2\phi) & 0 \\ 0 & a^2(t)(1-2\psi)\delta_{ij} \end{pmatrix}$$

ϕ and ψ are scalar potentials respectively, with ψ is the gravitational potential of classical Newtonian gravity. No tensor perturbation [8].

2.2. Field equations of general relativity

The dynamical equations of general relativity, namely the Einstein Field equations, are of the form:

$$8\pi G T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

The left side represents the distribution of matter or field, including the distribution of mass, energy, momentum, etc. The quantity on the right represents the geometric properties of space-time, describing the strength and curvature of the gravitational field. Among them, $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, and $T_{\mu\nu}$ is the energy-momentum tensor.

And the Ricci tensor is represented by the Christoffel symbol $\Gamma^{\lambda}_{\mu\nu}$

Equation:

$$R_{\mu\nu} = \partial_\lambda \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu\lambda} + \Gamma^{\lambda}_{\mu\nu}\Gamma^{\sigma}_{\lambda\sigma} - \Gamma^{\sigma}_{\mu\lambda}\Gamma_{\nu\sigma}$$

with $\Gamma^{\lambda}_{\mu\nu}$ is:

$$\Gamma^{\lambda}_{\mu\nu} = \tfrac{1}{2} g^{\lambda\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right)[9]$$

When expanding the disturbance, these quantities also need to be expanded order by order. In the following chapters, we will continue to discuss the specific forms after expansion and how to derive the wave equation of gravitational waves.

2.3. Transverse-traceless projection

In order to extract the effective source term of pure gravitational waves from these generalized perturbations, the transverse-traceless (TT) projection method must be used.

The transverse-traceless condition is an inherent constraint of gravitational waves, which requires the tensor perturbation to satisfy:

Transverse:

$$\nabla^i h^{(T)}_{ij} = 0$$

Traceless:

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$$\delta^{ij}h_{ij}^{(T)}=0$$

Specifically, for the term S, its TT projection form is:

$$S_{ij}^{TT}\left(\overrightarrow{k}\right)=\left(P_{il}P_{jm}-\tfrac{1}{2}P_{ij}P_{lm}\right)S_{lm}\left(\overrightarrow{k}\right)[10]$$

In this way, the evolution equation of second-order gravitational waves can be fully expressed as:

$$\ddot{h}_{ij}^{(T)}+3H\dot{h}_{ij}^{(T)}-\tfrac{1}{a^2}\nabla^2h_{ij}^{(T)}=S_{ij}^{TT}[11]$$

3. Computation of gravitational waves induced by second-order scalar perturbations

We choose the FLRW universe as the background.

The right side of the field equation describes the geometry of spacetime, so first perform a second-order expansion of the Ricci tensor:

$$R_{\mu\nu}=R_{\mu\nu}^{(0)}+R_{\mu\nu}^{(1)}+R_{\mu\nu}^{(2)}$$

For $R_{\mu\nu}^{(2)}$, the Ricci tensor is related to the Christoffel symbol, so taking the second order and substituting it into the Ricci tensor gives:

$$\partial_\lambda\Gamma_{\mu\nu}^{\lambda}=\partial_\lambda\left(\Gamma_{\mu\nu}^{(0)\lambda}+\delta\Gamma_{\mu\nu}^{(1)\lambda}+\delta\Gamma_{\mu\nu}^{(2)\lambda}\right)$$

Only take the second-order contribution $\partial_\lambda\delta\Gamma_{\mu\nu}^{(2)\lambda}$;

$$\partial_\nu\Gamma_{\mu\lambda}^{\lambda}=-\partial_\nu\left(\Gamma_{\mu\lambda}^{(0)\lambda}+\delta\Gamma_{\mu\lambda}^{(1)\lambda}+\delta\Gamma_{\mu\lambda}^{(2)\lambda}\right)$$

Only take the second-order contribution $\partial_\nu\delta\Gamma_{\mu\nu}^{(2)\lambda}$;

$$\Gamma_{\lambda\sigma}^{\lambda}\Gamma_{\mu\nu}^{\sigma}=\left(\Gamma_{\lambda\sigma}^{(0)\lambda}+\delta\Gamma_{\lambda\sigma}^{(1)\lambda}+\delta\Gamma_{\lambda\sigma}^{(2)\lambda}\right)\left(\Gamma_{\mu\nu}^{(0)\sigma}+\delta\Gamma_{\mu\nu}^{(1)\sigma}+\delta\Gamma_{\mu\nu}^{(2)\sigma}\right)$$

In theory, there are two second-order contribution, $\Gamma^{(0)}\cdot\delta\Gamma^{(2)}$ and $\Gamma^{(1)}\cdot\delta\Gamma^{(1)}$, however, The FLRW is spatially homogeneous and isotropic, so its Christoffel symbol only involves the background quantity. Therefore, when it is multiplied by $\delta\Gamma^{(2)}$, no non-linear combination of scalar perturbations is generated [12].

Only take the second-order contribution $\delta\Gamma_{\lambda\sigma}^{(1)\lambda}\cdot\delta\Gamma_{\mu\nu}^{(1)\sigma}$;

$$\Gamma_{\mu\sigma}^{\lambda}\Gamma_{\lambda\nu}^{\sigma}=\left(\Gamma_{\mu\sigma}^{(0)\lambda}+\delta\Gamma_{\mu\sigma}^{(1)\lambda}+\delta\Gamma_{\mu\sigma}^{(2)\lambda}\right)\left(\Gamma_{\lambda\nu}^{(0)\sigma}+\delta\Gamma_{\lambda\nu}^{(1)\sigma}+\delta\Gamma_{\lambda\nu}^{(2)\sigma}\right)$$

Only take the second-order contribution $\delta\Gamma_{\lambda\sigma}^{(1)\lambda}\cdot\delta\Gamma_{\mu\nu}^{(1)\sigma}$.

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Combine theses terms, get:

$$\delta R^{(2)}_{\mu\nu} = \partial_\lambda \delta\Gamma^{(2)\lambda}_{\mu\nu} - \partial_\nu \delta\Gamma^{(2)\lambda}_{\mu\lambda} + \delta\Gamma^{(1)\lambda}_{\mu\nu}\delta\Gamma^{(1)\sigma}_{\lambda\sigma} - \delta\Gamma^{(1)\sigma}_{\mu\lambda}\delta\Gamma^{(1)\lambda}_{\nu\sigma}$$

The Ricci tensor expanded to second order has been derived, and it is evident that both first- and second-order Christoffel symbols contribute to its structure. To incorporate the scalar potential into the perturbed metric, explicit forms of these Christoffel symbols are also required.

After obtaining the second-order form of the Ricci tensor, it becomes clear that contributions arise from both the first- and second-order Christoffel symbols. Accordingly, determining these terms is necessary for inserting the scalar potential into the metric.

A similar procedure used in deriving the second-order Ricci tensor is applied to the perturbed metric, allowing us to isolate the contributions from first- and second-order Christoffel symbols. This leads to the following expressions:

$$\delta\Gamma^{(1)\lambda}_{\mu\nu} = \frac{1}{2}\bar{g}^{\lambda\sigma}\left(\partial_\mu h^{(1)}_{\sigma\nu} + \partial_\nu h^{(1)}_{\sigma\mu} - \partial_\sigma h^{(1)}_{\mu\nu}\right)$$

$$\delta\Gamma^{(2)\lambda}_{\mu\nu} = \frac{1}{2}\bar{g}^{\lambda\sigma}\left(\partial_\mu h^{(2)}_{\sigma\nu} + \partial_\nu h^{(2)}_{\sigma\mu} - \partial_\sigma h^{(2)}_{\mu\nu}\right) - \frac{1}{2}h^{(1)\lambda}_{\ \ \sigma}\left(\partial_\mu h^{(1)}_{\sigma\nu} + \partial_\nu h^{(1)}_{\sigma\mu} - \partial_\sigma h^{(1)}_{\mu\nu}\right)$$

The curvature of space is mainly determined by the spatial and time derivatives of the spatial metric components, and these contributions are mainly represented by Γ^0_{ij}. By focusing on computing this component, we can significantly simplify the calculations while capturing all the necessary physical effects of second-order scalar perturbations on gravitational waves.

Also, computing all components leads to extremely complex expressions that are not only inefficient but also unnecessary for our specific purposes.

Substitute $\lambda = 0,\ \mu = j,\ v = i$ into $\delta\Gamma^{(1)\lambda}_{\mu\nu}, \delta\Gamma^{(2)\lambda}_{\mu\nu}$:

$$\delta\Gamma^{(1)\lambda}_{ij} = \frac{1}{2}\bar{g}^{\lambda\sigma}\left(\partial_i h^{(1)}_{0j} + \partial_j h^{(1)}_{0i} - \partial_0 h^{(1)}_{ij}\right)$$

$$\delta\Gamma^{(2)\lambda}_{ij} = \frac{1}{2}\bar{g}^{\lambda\sigma}\left(\partial_i h^{(2)}_{0j} + \partial_j h^{(2)}_{0i} - \partial_0 h^{(2)}_{ij}\right) - \frac{1}{2}h^{(1)0}_{\ \ 0}\left(\partial_i h^{(1)}_{0j} + \partial_j h^{(1)}_{0i} - \partial_0 h^{(1)}_{ij}\right)$$

(Since the FLRW background is selected, σ needs to be equal to 0)

Recall the FLRW metric in Newton gauge, we can find:

$$\bar{g}^{00} = -1, \bar{g}^{0i} = 0, \bar{g}^{ij} = \frac{\delta^{ij}}{a^2}$$

$$h^{(1)}_{00} = -2\psi, h^{(1)}_{0i} = 0, h^{(1)}_{ij} = -2a^2\psi\,\delta_{ij}$$

$$h^{(2)}_{00} = -2\psi^2, h^{(2)}_{0i} = 0, h^{(2)}_{ij} = -2a^2\psi^2\,\delta_{ij}$$

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[Due to the selected FLRW spacetime, assuming that there is no significant anisotropic stress in the universe, the two potential functions naturally converge, that is, $\phi = \psi$[13].

Then, substitute in $\delta\Gamma_{ij}^{(1)0}$:

$$\delta\Gamma_{ij}^{(1)0} = -\frac{1}{2}\partial_0 h_{ij}^{(1)} = -\frac{1}{2}\partial_0\left(-2a^2\psi\,\delta_{ij}\right) = \frac{1}{2}\times 2\,\delta_{ij}\,\partial_0\left(a^2\psi\right) = \delta_{ij}\left(2a\dot{a}\psi + a^2\dot{\psi}\right)$$

In $\delta\Gamma_{ij}^{(1)k}$:

$$\delta\Gamma_{ij}^{(1)k} = -\left(\partial_i\psi\,\delta_j^k + \partial_j\psi\,\delta_i^k - \delta_{ij}\partial^k\psi\right)$$

In $\delta\Gamma_{ij}^{(2)0}$:

$$\delta\Gamma_{ij}^{(2)0} = 2a^2\delta_{ij}\dot{\psi}\left(\psi + \phi\right)$$

In $\delta\Gamma_{ij}^{(2)k}$:

$$\delta\Gamma_{ij}^{(2)k} = -\left(\delta_j^k\partial_i\psi^2 + \delta_i^k\partial_j\psi^2 - \delta_{ij}\partial^k\psi^2\right) + 2a^2\psi\left(\delta_j^k\partial_i\psi + \delta_i^k\partial_j\psi - \delta_{ij}\partial^k\psi\right)$$

All the Christoffel symbols have been obtained and can be substitute into the Ricci tensor:

$$\delta R_{\mu\nu}^{(2)} = \partial_\lambda\delta\Gamma_{\mu\nu}^{(2)\lambda} - \partial_\nu\delta\Gamma_{\mu\lambda}^{(2)\lambda} + \delta\Gamma_{\mu\nu}^{(1)\lambda}\delta\Gamma_{\lambda\sigma}^{(1)\sigma} - \delta\Gamma_{\mu\lambda}^{(1)\sigma}\delta\Gamma_{\nu\sigma}^{(1)\lambda}$$

Using Mathematica to calculate, after calculating the time derivative and the space derivative, and then substituting and simplifying, get:

$$\delta^{(2)}R_{ij} =$$

$$2\delta_{ij}\frac{d}{dt}\left(a\dot{a}\psi^2 + a^2\psi\dot{\psi} - \varphi\left(2a\dot{a}\psi + a^2\dot{\psi}\right)\right) + \partial_i\partial_j\left(\psi^2\right) + \delta_{ij}\nabla^2\left(\psi^2\right) + 4\partial_i\psi\partial_j - 4\delta_{ij}(\nabla\psi)^2$$

The scalar potential function has appeared explicitly in the Ricci tensor. However, the Ricci tensor only describes how space-time is curved, but does not directly show gravitational waves. Therefore, the Transverse-Traceless Projection is needed to extract the source term.

For the projection matrix P:

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Using Mathematica for matrix and symbol operations, the source term is obtained:

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$$S_{ij} = \begin{pmatrix} S_{xx} & S_{xy} & 0 \\ S_{yx} & S_{yy} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$S_{xx} = 2\,\partial_x^2\left(\psi^2\right) + 4(\partial_x\psi)^2 - 2\left(\partial_x^2 + \partial_y^2\right)\left(\psi^2\right) - 4\left((\partial_x\psi)^2 + (\partial_y\psi)^2 + (\partial_z\psi)^2\right)$$

$$S_{yy} = 2\,\partial_y^2\left(\psi^2\right) + 4(\partial_y\psi)^2 - 2\left(\partial_x^2 + \partial_y^2\right)\left(\psi^2\right) - 4\left((\partial_x\psi)^2 + (\partial_y\psi)^2 + (\partial_z\psi)^2\right)$$

$$S_{xy} = S_{yx} = 2\,\partial_x\partial_y\left(\psi^2\right) + 4\left(\partial_x\psi\right)\left(\partial_y\psi\right)$$

It can be found that the potential function appears again in the source term, which shows that scalar perturbations can induce gravitational waves at the second order, and it is perpendicular to the propagation direction - only the x-y plane term can be left.

4. Conclusion

This paper systematically discusses the theoretical framework and calculation method of gravitational waves induced by second-order scalar perturbations. By selecting the FLRW metric under the Newtonian gauge, the second-order Christoffel symbol and Ricci tensor are derived, and the original source term is constructed through these. In order to extract the physical content directly related to gravitational waves, the TT projection is further applied to obtain the effective source term S^{TT} of the pure tensor mode. This shows that even if the initial perturbation of the universe only contains a scalar part, the nonlinear gravitational effect can still produce tensor perturbations in the second order, thereby forming gravitational waves.

This secondary gravitational wave induced by scalar perturbations provides important theoretical support for our understanding of the structure formation and evolution process of the early universe, and is expected to reveal more evolutionary information of the early universe.

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Appendix

```
a = a[t];

phi = phi[x, y, z, t];

psi = psi[x, y, z, t];

coords = \{x, y, z\};

delta = KroneckerDelta;

gradPsi2 = Grad[psi\^2, coords];

laplacianPsi2 = Div[Grad[psi\^2, coords], coords];

gradPsi = Grad[psi, coords];

dotGradPsi = gradPsi . gradPsi;

timeDerivativeTerm =

2 delta[i, j] D[a D[a, t] psi\^2 + a\^2 psi D[psi, t] -

phi (2 a D[a, t] psi + a\^2 D[psi, t]), t];

spaceSecondDerivativeTerm = D[psi\^2, \{coords[[i]], 1\}, \{coords[[j]], 1\}];

gradientSquareTerm = 4 D[psi, coords[[i]]] D[psi, coords[[j]]];

deltaRij = timeDerivativeTerm + spaceSecondDerivativeTerm +

delta[i, j] laplacianPsi2 + gradientSquareTerm -

4 delta[i, j] dotGradPsi;

deltaRijSimplified = Simplify[Expand[deltaRij]]

deltaRijSimplified

a = a[t];

phi = phi[x, y, z, t];

psi = psi[x, y, z, t];
```

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```
coords = \{x, y, z\};
(*Kronecker Delta*)
delta[i\_, j\_] := KroneckerDelta[i, j];
gradPsi = Grad[psi, coords];
gradPsi2 = Grad[psi\^2, coords];
laplacianPsi2 = Div[Grad[psi\^2, coords], coords];
timeDerivativeTerm =
2 delta[i, j] D[
a D[a, t] psi\^2 + a\^2 psi D[psi, t] -
phi (2 a D[a, t] psi + a\^2 D[psi, t]), t];
spaceSecondDerivativeTerm =
D[psi\^2, \{coords[[i]], 1\}, \{coords[[j]], 1\}];
gradientSquareTerm = 4 D[psi, coords[[i]]] D[psi, coords[[j]]];
(*gradient squared total*)
gradPsiSquare = gradPsi . gradPsi;
deltaRij =
timeDerivativeTerm + spaceSecondDerivativeTerm +
delta[i, j] laplacianPsi2 + gradientSquareTerm -
4 delta[i, j] gradPsiSquare;
P[vec\_] := IdentityMatrix[3] - Outer[Times, vec, vec]/(vec . vec);
propagationDirection = \{0, 0, 1\};
Pij = P[propagationDirection];
Sij = Simplify[Pij . deltaRij . Transpose[Pij]];
SijExpanded = Expand[Sij];
SijExpanded
P = \{\{1, 0, 0\}, \{0, 1, 0\}, \{0, 0, 0\}\};
```

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```
Rij = gradientSquareTerm + spaceSecondDerivativeTerm +
timeDerivativeTerm + (laplacianPsi2 -
4 gradPsi . gradPsi) KroneckerDelta[i, j];
Sij = Simplify[P . Rij . Transpose[P]]
MatrixForm[Sij]
```