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Stirling's Formula: Derivation and Applications

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Abstract. In this paper, we take a closer look at Stirling's formula, a method used to estimate factorials, particularly when nnn is large. Starting with its historical background, we then give a derivation using the Gamma function and examine how the formula behaves asymptotically. While it's a classic result in mathematics, we also discuss where it shows up in real-world problems—like signal processing, rubber-based materials, and even brain imaging. The way it links with Fourier transforms of the Gamma function shows how useful it is for interpreting patterns like power-law decay and frequency changes in signals.

Keywords: Stirling's approximation, Gamma function, asymptotic analysis, Fourier transform, signal processing, viscoelasticity, power-law decay, biomedical applications.

1. Introduction

1.1. The history

Stirling's formula, which is an important approximated tool for factorials and was introduced by James Stirling, who is a famous Scottish mathematician. This approximation gives a high efficient way to estimate factorials n! The positive number of variables that make it a good fit (the factorial of a positive integer n) using logarithmic and exponential functions. Because of its precise approximation for large values of n, the formula is widely used in many mathematic fields like probability theory and statistics.

The original form of Stirling's formula was introduced in Stirling's1730 treatise Methodus Differentialis, which use a logarithmic estimate for n!

$$\mathrm{ln(n!) \approx nlnn - n.}$$

After that, this first result was improved to formula with higher- order correction terms, which makes a more accurate estimate for factorials

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \ \ as\ n \to \infty$$

The addition of the $\sqrt{2\pi n}$ factor owed to Abraham de Moivre,, a current of Stirling who was working on probability theory about Stirling's formula, especially about the approximation of

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binomial coefficients for large values n

The appearance of Stirling's approximation was closely related to two mathematical advances during the 18^{th} century: logarithmic theory developments and the Gamma function introduction,which expand factorial operations to non-integer values. Euler played an important role by his investigations of the Gamma function, with Gauss later providing more accurate and precise mathematical foundations for these asymptotic expressions.

During the derivation process, Wallis's infinite product expression was used. Stirling's innovative method constructed a continuous form, revealing an unexpected relationship between discrete factorial operations and transcendental numbers. This relationship astonished the mathematicians of that time. This breakthrough demonstrated the significant characteristics of mathematics in the Enlightenment era, namely the profound connection between continuous analysis and discrete mathematics.

In the 19th and 20th centuries, the application scope of this formula expanded significantly, covering various fields such as information theory, computational complexity analysis, statistical mechanics (especially in Boltzmann's entropy formula), and so on. Even today, this formula remains an important tool in asymptotic analysis, focusing on improving the accuracy of error estimation and conducting more extensive expansions.

1.2. Overview

Stirling's formula serves as an approximation approach for factorials when the numbers involved become large. Although de Moivre was the first person who introduced a similar idea, the formula is more commonly relating to Stirling.Instead of relying on precise calculations, this method offers a practical way to make approximation to factorials, which proves useful in areas such as combinatorics, statistical mechanics, and probability. In this paper, we not only explore the derivation of Stirling's formula but also reveal the connection between Stirling's formula and the Fourier transform.

2. Derivation

2.1. Background knowledge

2.1.1. Proof of integral convergence

The Gamma function is defined by the integral:bounded above: $0 < t^{x-1}e^{-t} < t^{x-1}, t > 0$

$$\int_\epsilon^1 t^{x-1}dt = \left[\frac{1}{x}t^x\right]_\epsilon^1 = \frac{1}{x}\left(1 - \varepsilon^x\right) \quad \epsilon > 0$$

If x> 0

$$\int_\epsilon^1 t^{x-1}dt = \left[\frac{1}{x}t^x\right]_\epsilon^1 = \frac{1}{x}$$

Notice that $\int_0^1 t^{x-1}dt = \infty$ if x<0

By comparison $\int_0^1 t^{x-1}e^{-t}dt$ is finite

STEP2

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$$\int_1^{\infty} t^{x-1}e^{-t}dt = \lim_{a\to\infty}\int_1^{a} t^{x-1}e^{-t}dt$$

When t is large, $\int_0^1 t^{x-1}e^{-t}dt$ is small. Because e−t decreases much more rapidly than tx−1 increases.

Precise: Maclaurin serves for e^t

$$e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!}$$

Pick n large enough : n ≥x + 1

$$e^t > \frac{t^n}{n!} \Rightarrow e^{-t} < n!t^{-n}$$

$$t^{x-1}\cdot e^{-t} < n!t^{x-n-1} \leq n!t^{-2}$$

$$\int_1^{\infty} n!t^{-2}dt = n!\lim_{\alpha\to\infty}\int_1^{\alpha} t^{-2}dt = n!\lim_{a\to\infty}\left[-\frac{1}{t}\right]_1^a = n!\ l_{\alpha\to\infty}\left(1-\frac{1}{a}\right)n! < \infty$$

Hence $\int_1^{\infty} t^{x-1}e^{-t}dt < \infty$ Γ(t) converges for every x> 0.

2.1.2. Γ(x) characterization

Γ(x) is uniquely characterized by: for x > 0

(1) Γ(1) =1 (2) Γ(x + 1) = x Γ(x) (3) Γ(x) is log convex

2.1.3. Proof of γ(x)

Proof of (1):

Through the definition of the Gamma function, we have:

$$\Gamma(1) = \int_0^{\infty} t^{1-1}e^{-t}dt.$$

Since 1 −1 =0, this simplifies to:

$$\Gamma(1) = \int_0^{\infty} t^{0}e^{-t}dt.$$

Since t^0 =1, we obtain:

$$\Gamma(1) = \int_0^{\infty} e^{-t}dt.$$

This is a standard integral, which evaluates to:

$$\int_0^{\infty} e^{-t}dt = 1.$$

Thus, we conclude:

Γ(1)=1

Proof of (2):

Let x∈C with Re(x)> 0.The Gamma function is defined by:

$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt$

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We aim to prove the recurrence relation:

$\Gamma(x + 1) = x\,\Gamma(x)$

We begin by writing:

$\Gamma(x + 1) = \int_0^\infty t^x e^{-t} dt$

Apply integration by parts. Let:

$$u = t^x \Rightarrow du = xt^{x-1}dt, dv = e^{-t}dt \Rightarrow v = -e^{-t}$$

Then:

$$\Gamma\left(x + 1\right) = \left[-t^x e^{-t}\right]_0^\infty + \int_0^\infty xt^{x-1}e^{-t}dt$$

We now analyze the boundary word $\left[-t^x e^{-t}\right]_0^\infty$

As $t \to \infty$:

$t^x e^{-t} = e^{x\log t - t} \to 0$ since exponential decay dominates

As $t \to 0^+$:

$t^x e^{-t} \sim t^x \to 0$ since Re(x)> 0

Therefore:

$$\left[-t^x e^{-t}\right]_0^\infty = 0$$

Thus:

$$\Gamma\left(x + 1\right) = x\int_0^\infty t^{x-1}e^{-t}dt = x\Gamma\left(x\right)$$

$$\Gamma(x + 1) = x\Gamma(x)$$

Proof of (3)

The first step in establishing log-convexity is to calculate the derivatives of $\ln\Gamma(x)$:

[DiGamma and TriGamma Functions] The logarithmic derivative of the Gamma function which is known as the diGamma function ψ(x), and its derivative, the triGamma function ψ′(x), are given by:

$$\begin{aligned} \psi(x) &= \frac{d}{dx}\ln\Gamma\left(x\right) = \frac{\Gamma'(x)}{\Gamma(x)} \\ \psi'(x) &= \frac{d^2}{dx^2}\ln\Gamma\left(x\right) = \frac{\Gamma''(x)}{\Gamma(x)} - \left(\frac{\Gamma'(x)}{\Gamma(x)}\right)^2 \end{aligned}$$

Using the integral definition of $\Gamma(x)$， we can express these derivatives as:

$$\begin{aligned} \Gamma'(x) &= \int_0^\infty t^{x-1}e^{-t}\ln t dt \\ \Gamma''(x) &= \int_0^\infty t^{x-1}e^{-t}\left(\ln t\right)^2 dt \end{aligned}$$

This gives us the following expressions:

$$\begin{aligned} \psi(x) &= \frac{\int_0^\infty t^{x-1}e^{-t}\ln t dt}{\int_0^\infty t^{x-1}e^{-t}dt} \\ \psi'(x) &= \frac{\int_0^\infty t^{x-1}e^{-t}\left(\ln t\right)^2 dt}{\int_0^\infty t^{x-1}e^{-t}dt} - \left(\frac{\int_0^\infty t^{x-1}e^{-t}\ln t dt}{\int_0^\infty t^{x-1}e^{-t}dt}\right)^2 \end{aligned}$$

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The non-negativity of $\psi'(x)$ follows from the Cauchy-Schwarz inequality. Consider the inner product space of measurable functions on $(0,\infty)$ with weight function $t^{x-1}e^{-t}$

$$\langle f,g\rangle = \int_0^\infty f(t)g(t)t^{x-1}e^{-t}dt$$

Letting f(t) = lnt , g(t) =1, the Cauchy-Schwarz inequality show:

$$\langle f,g\rangle^2 \leq \langle f,f\rangle\langle g,g\rangle$$

Which translates to:

$$\left(\int_0^\infty t^{x-1}e^{-t}lntdt\right)^2 \leq \left(\int_0^\infty t^{x-1}e^{-t}\left(lnt\left(\int_0^\infty t^{x-1}e^{-t}dt\right)\right.\right.$$

Dividing both sides by $\left(\int_0^\infty t^{x-1}e^{-t}\mathrm{lntdt}\right)^2$ gives exactly
$\psi'(x)\geq 0$
Since $\psi'(x) = \frac{d^2}{dx^2}ln\Gamma(x) \geq 0$ for all x>0， we conclude that lnΓ(x)is convex， and therefore Γ(x)is logarithmically convex on $(0,\infty)$.

2.2. Derivation of stirling's formula

$$\left(1+\frac{1}{k}\right)^k < e < \left(1+\frac{1}{k}\right)^{k+1}$$

$$\prod_{k=1}^{n-1}\left(\frac{k+1}{k}\right)^k < e^{n-1} < \prod_{k=1}^{n-1}\left(\frac{k+1}{k}\right)^{k+1}$$

$$\mathrm{LHS}: \left(\frac{2}{1}\right)^1\cdot\left(\frac{3}{2}\right)^2\cdot\left(\frac{4}{3}\right)^3\cdots\left(\frac{n}{n-1}\right)^{n-1} = \frac{n^{n-1}}{(n-1)!}$$

$$\mathrm{RHS}: \left(\frac{2}{1}\right)^2\cdot\left(\frac{3}{2}\right)^3\cdot\left(\frac{4}{3}\right)^4\cdots\left(\frac{n}{n-1}\right)^{n} = \frac{n^{n}}{(n-1)!}$$

$$\text{so } \frac{n^{n-1}}{(n-1)!} < e^{n-1} < \frac{n^n}{(n-1)!}$$

Rearrange and multiply by n

$$en^ne^{-n} < n! < en^{n+1}e^{-n}$$

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, x>0$$

$$\Gamma(x) = (x-1)!$$

$(x-1)!$ which is Γ(x) should be between $x^{x-1}e^{-x}$ and x^xe^{-x}
let $f(x) = x^{x-\frac{1}{2}}e^{-x+\mu(x)}$
whereμ(x) is an error term and $x^{x-\frac{1}{2}}e^{-x}$ is halfway between $x^{x-1}e^{-x}$ and x^xe^{-x}
We'd like f(x) to satisfy 2) and 3), because then it would have to be a multiple of Γ(x)! Let's calculate:

$$x = \frac{f(x+1)}{f(x)} = \frac{(x+1)^{x+\frac{1}{2}}e^{-x-1}e^{\mu(x+1)}}{x^{x-\frac{1}{2}}e^{-x}e^{\mu(x)}}$$

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$$x = x\left(1+\frac{1}{x}\right)^x + \frac{1}{2}e^{-1}e^{\mu(x+1)-\mu(x)}$$

Take in:

$$\mu(x) - \mu(x+1) = \left(x+\frac{1}{2}\right)\ln\left(1+\frac{1}{x}\right) - 1$$

Let g(x) =RHS

$$\mu(x) = \sum_{n=0}^{\infty} g(x+n)$$

To establish it is convex , we have to find an upper bound.
If $|x| < 1$, ,then apply Taylor series:

$$\ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \ldots$$

$$\ln(1-y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \cdots$$

so $\ln\left(\frac{1+y}{1-y}\right) = 2\left(y + \frac{y^3}{3} + \frac{y^5}{5} + \cdots\right)$

$$\text{set } y = \frac{1}{2x+1}$$

$$\ln\left(\frac{1+y}{1-y}\right) = \ln\left(\frac{2x+2}{2x}\right) = \ln\left(1+\frac{1}{x}\right)$$

$$\text{so } \frac{1}{2}\ln\left(1+\frac{1}{x}\right) = \frac{1}{2x+1} + \frac{1}{3(2x+1)^3} + \frac{1}{5(2x+1)^5} + \cdots$$

$$g(x) = \left(x+\frac{1}{2}\right)\ln\left(1+\frac{1}{x}\right) - 1 = \frac{1}{(2x+1)^2}\left(\frac{1}{3} + \frac{1}{5(2x+1)^2} + \frac{1}{7(2x+1)^4} + \cdots\right)$$

$$\text{so } 0 < g(x) < \frac{1}{3(2x+1)^2}\left(1 + \frac{1}{(2x+1)^2} + \frac{1}{(2x+1)^4} + \cdots\right)$$

$$\left(1 + \frac{1}{(2x+1)^2} + \frac{1}{(2x+1)^4} + \cdots\right) = \frac{1}{1-\frac{1}{(2x+1)^2}} = \frac{(2x+1)^2}{4x(x+1)}$$

$$0 < g(x) < \frac{1}{12}\left(\frac{1}{x} - \frac{1}{x+1}\right), x > 0$$

$$0 < \sum_{n=0}^{\infty} g(x+n) < \mu(x) = \sum_{n=1}^{\infty}\left(\frac{1}{x} - \frac{1}{x+1}\right) = \frac{1}{12x}$$

$$0 < \mu(x) < \frac{1}{12x}$$

It means these series converge
So with this choice of μ(x), the function $f(x) = x^{x-\frac{1}{2}}e^{-x}e^{\mu(x)}$ satisfies the factorial property
$f(x+1) = xf(x)$
So we want to prove the function

$$f(x) = x^{x-\frac{1}{2}}e^{-x+\mu(x)}$$

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,where $\mu(x) = \sum_{n=0}^{\infty} g(n)$ and $g(x) = (x + \frac{1}{2})\ln(1 + \frac{1}{x})$
is log convex. We have

$$\ln f(x) = (x - \frac{1}{2})\ln x - x + \mu(x).$$

We want to show that ln f(x) is convex.

Since the second derivative is 0, the total of the convex functions is convex (because the latter is defined by convexity).

1. –x is convex, as its second derivative is 0.

2. For $(x - \frac{1}{2})\ln x$

$$[(x - \frac{1}{2})\ln x]' = \ln x + (x - \cdot \frac{1}{x} = \ln x + 1 - \frac{1}{2x}.$$

The second derivative is:

$$[\ln x + 1 - \frac{1}{2x}]' = \frac{1}{x} + \frac{1}{2x^2} > 0 for x > 0.$$

3.$\mu(x)$ is the sum of translates of g(x)， so it is convex if g(x) is convex. Proof that $g''(x)>0$ for $x>0$ Let

$$g(x) = (x + \frac{1}{2})\ln(1 + \frac{1}{x}) - 1.$$

Using the product rule:

$$g'(x) = \ln(1 + \frac{1}{x}) + (x + \frac{1}{2}) \cdot \frac{d}{dx}[\ln(1 + \frac{1}{x})].$$

Compute the derivative:

$$\frac{d}{dx}\left[\ln\left(1 + \frac{1}{x}\right)\right] = \frac{-1/x^2}{1+1/x} = -\frac{1}{x(x+1)}$$

Thus:

$$g'\left(x\right) = ln\left(1 + \frac{1}{x}\right) - \frac{x+\frac{1}{2}}{x(x+1)}.$$

Differentiate $g'(x)$:

$$g''\left(x\right) = \frac{d}{dx}\left[ln\left(1 + \frac{1}{x}\right) - \frac{d}{dx}\left[\frac{x+\frac{1}{2}}{x(x+1)}\right].$$

The first term is:

$$\frac{d}{dx}ln\left(1 + \frac{1}{x}\right)\right] = -\frac{1}{x(x+1)}.$$

For the second term,use the quotient rule:

$$\frac{d}{dx}\left[\frac{x+\frac{1}{2}}{x(x+1)}\right] = \frac{(1)\cdot x(x+1) - (x+\frac{1}{2})(2x+1)}{x^2(x+1)^2}.$$

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Expand the numerator:

$$x(x+1)-\left(x+\tfrac{1}{2}\right)(2x+1)=x^2+x-\left(2x^2+x+x+\tfrac{1}{2}\right)=-x^2-x-\tfrac{1}{2}.$$

Thus:

$$\frac{d}{dx}\left[\frac{x+\frac{1}{2}}{x(x+1)}\right]=\frac{-x^2-x-\frac{1}{2}}{x^2(x+1)^2}.$$

Now combine the terms:

$$g''(x)=-\frac{1}{x(x+1)}-\left(\frac{-x^2-x-\frac{1}{2}}{x^2(x+1)^2}\right)=-\frac{1}{x(x+1)}+\frac{x^2+x+\frac{1}{2}}{x^2(x+1)^2}.$$

Simplify:

$$g''(x)=\frac{-(x+1)\left(x^2+x+\frac{1}{2}\right)+\left(x^2+x+\frac{1}{2}\right)}{x^2(x+1)^2}=\frac{\frac{1}{2}}{x^2(x+1)^2}=\frac{1}{2x^2(x+1)^2}.$$

For all x> 0:

$$x^2>0,\ (x+1)^2>0$$

so:

$$g''(x)=\frac{1}{2x^2(x+1)^2}>0.$$

Thus it is log convex
But this means Γ(x) must be a multiple of f(x):

$$\Gamma(x)=\alpha f(x)=\alpha x^{x-\frac{1}{2}}e^{-x+\mu(x)}=\alpha x^{x-\frac{1}{2}}e^{-x+\frac{\theta}{12x}}.$$

so we have to compute α

$$y=x^n e^{-x}$$

By calculus.We find the maximum value at x = n by $y'=nx^{n-1}e^{-x}+x^n\cdot\left(-e^{-x}\right)$

$$y'=0$$

$$x^{n-1}e^{-x}(n-x)=0$$

$$x=n$$

Inflections points at $x=n+\sqrt{n}, x=n-\sqrt{n}$

$$y''=n(n-1)x^{n-2}\cdot e^{-x}+nx^{n-1}\cdot\left(-e^{-x}\right)+(-n)x^{n-1}e^{-x}+x^n e^{-x}=n(n-1)x^{n-2}e^{-x}-2nx^{n-1}e^{-x}+x^n e^{-x}$$

$$y''=0$$

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$$e^{-x}x^{n-2}\left[n(n-1)+x^2-2nx\right]=0$$

$$x^2-2nx+n(n-1)=0$$

$$x=\frac{2n\pm2\sqrt{n}}{2}=n\pm\sqrt{n}.$$

Because the shape of $y=x^n e^{-x}$ is similar to normal distribution analog $\mu\leftrightarrow n,\sigma\leftrightarrow\sqrt{n}$
So we change of variable $t=\frac{x-n}{\sqrt{n}}x=t\sqrt{n}+n$

$$\therefore n!=\int_0^\infty x^n e^{-x}dx=\int_{-\sqrt{n}}^\infty\left(n+t\sqrt{n}\right)^n e^{-(t\sqrt{n}+n)}\cdot\sqrt{n}dt.$$

$$=\frac{n^n\sqrt{n}}{e^n}\int_{-\sqrt{n}}^\infty\left(1+\frac{t}{\sqrt{n}}\right)^n e^{-\sqrt{n}t}dt$$

n! is equal to n!

$$\alpha=\int_{\sqrt{n}}^\infty\left(1+\frac{t}{\sqrt{n}}\right)^n e^{-\sqrt{n}t}dt$$

$$when\ n\to\infty,\ \text{for}\ each\ t\ \left(1+\frac{t}{\sqrt{n}}\right)^n e^{-\sqrt{n}t}$$

$$\to e^{-\frac{t^2}{2}}$$

$$\to\int_{-\infty}^\infty e^{-\frac{t^2}{2}}dt$$

$$\alpha=I=\int_{-\infty}^\infty e^{-\frac{t^2}{2}}dt$$

$$I^2=\int_{-\infty}^\infty e^{-\frac{t^2}{2}}dt\int_{-\infty}^\infty e^{-\frac{s^2}{2}}ds$$

$$t=r\cos\theta s=r\sin\theta$$

$$I^2=\int_0^{2\pi}\int_0^\infty e^{-\frac{r^2}{2}}rdrd\theta$$

$$I^2=\int_0^{2\pi}d\theta\int_0^\infty e^{-\frac{r^2}{2}}rdr$$

$$=2\pi$$

$$\alpha=I=\sqrt{2\pi}$$

$$\alpha=\sqrt{2\pi}$$

2.3. Theorem (stirling's formula)

For x>0:

$$\Gamma(x)=\sqrt{2\pi}x^{x-\frac{1}{2}}e^{-x+\frac{\theta}{12x}},0<\theta<1\ \text{depends on x}$$

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$$\mathrm{n}! = \sqrt{2\pi}\mathrm{n}^{\mathrm{n}+\frac{1}{2}}\mathrm{e}^{-\mathrm{n}+\frac{\theta}{12\mathrm{n}}}.$$

2.4. Stirling's approximation

When n is large:

$$n! \approx \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n} = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

3. Fourier transform

3.1. Fourier transform of $x^{z-1}e^{-x}$ and relation to the Gamma function

3.1.1. Corrected derivation

There is no direct correlation between the function $f(x) = x^{z-1}e^{-x}$ and the Gamma function $\Gamma(z)$, so the $f(x) = x^{z-1}e^{-x}$ should be related with the Fourier transform.

Here is the revised derivation:

3.1.2. Fourier transform of f(x)

The single-sided Fourier transform of f(x) is:

$$F\{f(x)\}(\omega) = \int_0^\infty x^{z-1}e^{-x}e^{-i\omega x}dx.$$

3.1.3. Simplification

Combine the exponential terms:

$$F\{f(x)\}(\omega) = \int_0^\infty x^{z-1}e^{-(1+i\omega)x}dx.$$

3.1.4. Connection to the Gamma function

Recognize this as a Gamma function with a modified parameter:

$$F\{f(x)\}(\omega) = \Gamma(z)\cdot(1+i\omega)^{-z}$$

3.1.5. Double integral interpretation

To directly write a double integral(if required)， relate the Gamma function and Fourth Fourier Transform:

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx \Longrightarrow F\{f(x)\}(\omega) = \int_0^\infty \left(\int_{-\infty}^\infty e^{-i\omega x}d\omega\right)x^{z-1}e^{-x}dx.$$

However， this is merely symbolic; the rigorous result is given by the closed-form expression

$$\Gamma(z)(1+i\omega)^{-z}$$

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4. Derivation of the complex fourier series

The Fourier series' complicated exponential form， $f(t)=\sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t}$， is derived below.

4.1. Trigonometric fourier series

A periodic function f(t) which has period T can be written in the form:

$$f(t)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos(n\omega_0 t)+b_n\sin(n\omega_0 t)\right)$$

Where $\omega_0=\frac{2\pi}{T}$ ， and the coefficient are

$$a_n=\frac{2}{T}\int_{-T/2}^{T/2} f\left(t\right)\cos\left(n\omega_0 t\right)dt, b_n=\frac{2}{T}\int_{-T/2}^{T/2} f\left(t\right)\sin\left(n\omega_0 t\right)dt.$$

4.2. Complex exponential conversion

Using the formula of Euler's, which $e^{ix}=cosx+isinx$, write the following trigonometric functions:

$$\cos\left(n\omega_0 t\right)=\frac{e^{in\omega_0 t}+e^{-in\omega_0 t}}{2}, \sin\left(n\omega_0 t\right)=\frac{e^{in\omega_0 t}-e^{-in\omega_0 t}}{2i}.$$

4.3. Combining terms

Substitute the exponential forms in to the trigonometric series:

$$f\left(t\right)=a_0+\sum_{n=1}^{\infty}\left(a_n\frac{e^{in\omega_0 t}+e^{-in\omega_0 t}}{2}+b_n\frac{e^{in\omega_0 t}-e^{-in\omega_0 t}}{2i}\right)$$

Combine coefficients into C_n :

$$C_0=a_0, C_n=\frac{a_n-ib_n}{2}, C_{-n}=\frac{a_n+ib_n}{2}\left(n\geq 1\right).$$

The final complex Fourier series becomes:

$$f\left(t\right)=\sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t}$$

With coefficients:

$$C_n=\frac{1}{T}\int_{-T/2}^{T/2} f\left(t\right)e^{-in\omega_0 t}dt$$

Assuming that the fundamental frequency is ω_0 , the Fourier series can be stated as follows; the Fourier series representation of a periodic function is as follows:

$$f\left(t\right)=\sum_{n=-\infty}^{\infty} C_n e^{in\omega_0 t} \tag{1}$$

Where the coefficients C_n are calculated as:

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$$C_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f\left(t\right)e^{-in\omega_0 t}dt \quad (2)$$

4.3.1. Extension to non-periodic functions

In the case of non-periodic functions, we consider them as the limiting case of periodic functions with $T \to \infty$. Combining Equations (1) and (2):

$$f\left(t\right) = \sum_{n=-\infty}^{\infty}\left[\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f\left(t\right)e^{-in\omega_0 t}dt\right]e^{in\omega_0 t} \quad (3)$$

Let $\omega = n\omega_0$, where as the causes are frequency interval becomes: $\Delta\omega = (n+1)\omega_0 - n\omega_0 = \omega_0$ Using the period-frequency relationship $T = \frac{2\pi}{\omega_0}$, Equation (3) can be rewritten as:

$$f\left(t\right) = \sum_{\omega=-\infty}^{\infty}\frac{\Delta\omega}{2\pi}\left[\int_{-\frac{T}{2}}^{\frac{T}{2}} f\left(t\right)e^{-i\omega t}dt\right]e^{i\omega t} \quad (3')$$

Rearranging constants and simplifying:

$$f\left(t\right) = \frac{1}{2\pi}\sum_{\omega=-\infty}^{\infty}\left[\int_{-\frac{T}{2}}^{\frac{T}{2}} f\left(t\right)e^{-i\omega t}dt\right]e^{i\omega t}\Delta\omega \quad (4)$$

As $T \to \infty$:

$$\Delta\omega \to d\omega, \sum_{\omega} \to \int_{-\infty}^{\infty}$$

The discrete summation in Equation(4) transforms into a continuous Riemann integral:

$$f\left(t\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f\left(t\right)e^{-i\omega t}dt\right]e^{i\omega t}d\omega \quad (5)$$

Thus,

$$F\left(\omega\right) = \int_{-\infty}^{\infty} f\left(t\right)e^{-i\omega t}dt$$

4.4. Gamma function and its fourier transform

4.4.1. Relationship between exponential frequency modulated signal and Gamma function

An exponential frequency modulated (FM) signal can be stated as follows:

$$s\left(t\right) = Ae^{j\phi(t)}$$

Where the phase $\phi(t)$ is given by:

$$\phi\left(t\right) = 2\pi f_0 \frac{e^{\alpha t}-1}{\alpha}$$

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The Fourier transform of an exponentially frequency modulated signal can be used in some circumstances to include integrals of the form:

$$\int_0^\infty e^{-ax}x^b dx$$

Which can be expressed using the Gamma function. The Gamma function is defined as:

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx, \mathrm{Re}(Z) > 0$$

For $s \in C$ with $\Re(s) > 0$:

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt, where\ t^{s-1} = e^{(s-1)\ln t}$$

4.5. Fourier transform of the Gamma function

We test the integral in order to calculate its Fourier transform:

$$F(\omega) = \int_0^\infty x^{z-1}e^{-x}e^{-i\omega x}dx$$

Combining the exponential terms:

$$F(\omega) = \int_0^\infty x^{z-1}e^{-(1+i\omega)x}dx$$

4.6. Variable substitution

Using the substitution $u = (1+i\omega)x$, the integral simplifies to

$$F(\omega) = (1+i\omega)^{-z}\Gamma(z)$$

4.6.1. Detailed substitution steps

Assume that $u = (1+i\omega)x$, then $du = (1+i\omega)dx$ or $dx = \frac{du}{1+i\omega}$ substituting the element in the integral:

$$F(\omega) = \int_0^\infty \left(\frac{u}{1+i\omega}\right)^{z-1} e^{-u} \cdot \frac{du}{1+i\omega}$$

$$F(\omega) = (1+i\omega)^{-z}\int_0^\infty u^{z-1}e^{-u}du = (1+i\omega)^{-z}\Gamma(z)$$

4.7. Asymptotic behavior for large z

For large z, Stirling's approximation provides:

$$\Gamma(z) \approx \sqrt{2\pi z}\left(\frac{z}{e}\right)^z$$

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5. Application in signal processing

5.1. Power-law decay in signals

The Fourier transform $F(\omega) = (1 + i\omega)^{-z}\Gamma(z)$ models power-law decay in signals.

For instance, in viscoelastic materials, this describes stress relaxation with a memory kernel proportional to t^{z-1} .

5.1.1. Example1: polymer melts (natural rubber)

For polyisoprene rubber, stress relaxation under step strain follows: $\sigma(t) \propto t^{z-1}, z \in (0,1)$. The memory kernel ' s Fourier transform matches the given expression [1].

5.1.2. Example2: automotive rubber dampers

Stress relaxation in car suspension components follows:

$$\sigma(t) = \sigma_0 \cdot E_z(-(t/\tau)^z)$$

Where $E_z(-(t/\tau)^z)$ is the Mittag-Leffler function [2].

5.1.3. Example3: chewing gum viscoelasticity

Experimental measurements show:

$$G(t) = G_0 \cdot (t/t_0)^{-0.3}$$

Matching the model when $z = 0.7$[3].

5.2. Communication systems

In order to improve the anti-interference ability and integrity of the signal, EFM signals are adopted in wireless communication. According to [4], in mobile communications, frequency modulated signals improve spectral efficiency and may reduce the impact of multi-path fading.

5.3. Radar systems

Radar systems use RFW (RF) wave forms to enhance target detection and resolution. This paper discusses the application of frequency-modulated continuous wave (FMCW) radar through [5], which often uses exponentially chirped signals to achieve better range resolution and clutter suppression.

5.4. Biomedical signal processing

In biomedical engineering, EFM signals are applied in medical imaging and neural signal analysis. Study [6] explores the use of frequency-modulated signals in functional magnetic resonance imaging (fMRI) to improve brain activity mapping.

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